# Lecture 20:

**Recall:** Laplacian masking : $g = f - \Delta f$ (Obtain a sharp image from a smooth image)

(non-smooth)

Conversely, to get a smooth image $f$ from a non-smooth image $g$, we can solve the PDE for $f$ : $-\Delta f + f = \underbrace{g}_{\text{known}}$

unknown

We will show that solving the above equation is equivalent to minimizing something :

$$E(f) = \iint \big(f(x,y) - g(x,y)\big)^2 \, dx\, dy + \iint \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 dx\, dy$$

In the discrete case, the PDE can be approximated (discretized) to get:

$$f(x,y) = g(x,y) + [f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)]$$

for all $(x,y)$ (Linear System)

<u>Remark:</u> Assume $f$ and $g$ are periodically extended.

$\nabla f + f = g$ can also be solved in the frequency domain:

$$DFT(f) = DFT(g + \underbrace{\Delta f}_{p * f})$$

$$\therefore DFT(f)(u,v) = DFT(g)(u,v) + c\,DFT(p)(u,v)\,DFT(f)(u,v)$$

$$\Leftrightarrow DFT(f)(u,v) = \left[\frac{1}{1 - c\,DFT(p)(u,v)}\right] DFT(g)(u,v)$$

$\downarrow$ inverse DFT

$f(x,y)$ !!

Consider: $E_{discrete}(f) = \sum_{x=1}^{N}\sum_{y=1}^{N}\left(f(x,y) - g(x,y)\right)^2 + \sum_{x=1}^{N}\sum_{y=1}^{N}\left[\left(f(x+1,y) - f(x,y)\right)^2 + \left(f(x,y+1) - f(x,y)\right)^2\right]$

Suppose $f$ is a minimizer of $E_{discrete}$. Then, for each $(x,y)$,

$E_{discrete}$ depends on $f(x,y)$ for each $(x,y)$

$$\frac{\partial E_{discrete}}{\partial f(x,y)} = 0.$$

$$= 2\left(f(x,y) - g(x,y)\right) + 2\left(f(x+1,y) - f(x,y)\right)(-1) + 2\left(f(x,y+1) - f(x,y)\right)(-1)$$
$$+ 2\left(f(x,y) - f(x-1,y)\right) + 2\left(f(x,y) - f(x,y-1)\right)$$

By simplification, we get:

$$f(x,y) = g(x,y) + \left[f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y)\right]$$

The continuous version of $E_{discrete}$ can be written as:

$$E(f) = \iint \left(f(x,y) - g(x,y)\right)^2 + \iint \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]}_{|\nabla f|^2} dx\,dy$$

## Remark:

- Solving $f = g + \Delta f$ is equivalent to energy minimization
- The first term in $E_{discrete}$ is called the fidelity term.
  Aim to find $f$ that is close to $g$.
- The second term is called the regularization term. Aim to enhance smoothness.

# Image processing by minimization.

Generally, we are given:

e.g.

$$E(\underbrace{f}_{f:[a,b]\times[a,b]\to\mathbb{R}}) = \int_a^b\int_a^b \left(f(x,y)-g(x,y)\right)^4 dxdy + \int_a^b\int_a^b |\nabla f| \, dxdy$$

$$\Big\downarrow$$

PDE

$$\Big\downarrow \text{ Discretize the PDE}$$

Recussive relationship

Suppose $f$ minimizes $E(f)$.

$(f : [a,b] \times [a,b] \to \mathbb{R})$

Fix $f$ and consider a random pertubation. $v : [a,b] \times [a,b] \to \mathbb{R}$

Then: consider $f + \varepsilon v : [a,b] \times [a,b] \to \mathbb{R}$

Define a function $S : \mathbb{R} \to \mathbb{R}$ by:

$$S(\varepsilon) = E(f + \varepsilon v)$$

Then: $\left.\dfrac{dS}{d\varepsilon}\right|_{\varepsilon=0} = 0 \quad \left(\because \; S(0) = E(f) = \text{minimum}\right)$